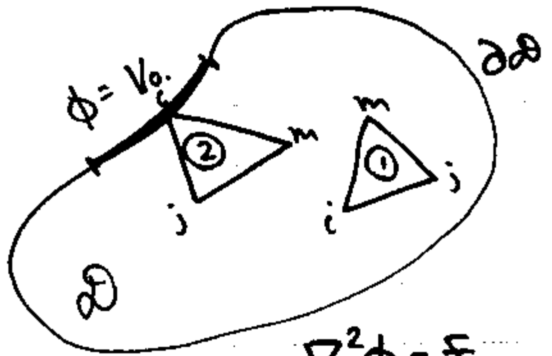
Assembling the Final System of Equations



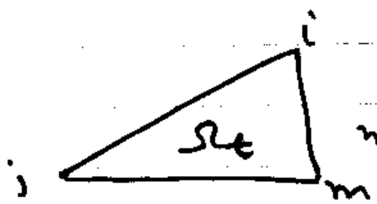
$$-\nabla^2\phi = f$$

① ijm interior triangle
- all nodes are free
- also for nodes on a Neuman B.C.

② ijm has a node on the Dirichlet B.C.

- these will contribute differently to the final matrix equation.

① contribution for the interior triangle



i, j, m node numbers

t - triangle #

$$F = \sum_{t=1}^{T} F_t$$

T is the number of triangles.

N is the # of nodes (free).

$$F_t = \int_{\Omega_t} (\nabla\phi)^2 \, dx\,dy - 2\int_{\Omega_t} f\phi \, dx\,dy$$

$$\phi(x,y) = \alpha_i(x,y)\phi_i + \alpha_j(x,y)\phi_j + \alpha_m(x,y)\phi_m$$

$\alpha_i, \alpha_j, \alpha_m$ are the shape functions

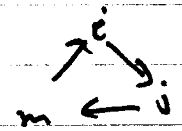
$$\bar\nabla\phi = \bar\nabla\alpha_i\,\phi_i + \bar\nabla\alpha_j\,\phi_j + \bar\nabla\alpha_m\,\phi_m$$

$$= \bar\nabla\underline{\alpha}^T\underline{\phi} = \underline{\phi}^T\bar\nabla\underline{\alpha}^T$$

$$\bar{\nabla}\underline{\alpha} = \begin{pmatrix} \bar{\nabla}\alpha_i \\ \bar{\nabla}\alpha_j \\ \bar{\nabla}\alpha_m \end{pmatrix} \qquad \underline{\phi} = \begin{pmatrix} \phi_i \\ \phi_j \\ \phi_m \end{pmatrix}$$

recall $\alpha_i = \dfrac{1}{2A_e}(a_i + b_i x + c_i y)$

$$a_i = (x_j y_m - x_m y_j)$$
$$b_i = (y_j - y_m)$$
$$c_i = (x_m - x_j)$$



$$\bar{\nabla}\alpha_i = \frac{1}{2A_e}\left[b_i \hat{x} + c_i \hat{y} \right]$$

$$F_t = \underline{\phi}^T \int_{\Omega_t} \bar{\nabla}\underline{\alpha}\, \bar{\nabla}\underline{\alpha}^T\, dx\, dy\ \underline{\phi} - 2\underline{\phi}^T \int_{\Omega_t} \underline{\alpha}\, F\, dx\, dy$$

$$= \underline{\phi}^T S^{(t)} \underline{\phi} - 2\,\underline{\phi}^T \underline{b}^{(t)}$$

$$S^{(t)} = \int_{\Omega_t} \bar{\nabla}\underline{\alpha}\, \bar{\nabla}\underline{\alpha}^T\, dx\, dy \qquad \underline{b}^{(t)} = \int_{\Omega_t} \underline{\alpha}\, F\, dx\, dy$$

recall:

$$\left(\frac{\partial y}{\partial \underline{a}} = \frac{\partial \underline{b}^T}{\partial \underline{a}} \underline{c} + \frac{\partial \underline{c}^T}{\partial \underline{a}} \underline{b} \right)$$

vector matrix

minimizing F_t:

$$0 = \frac{\partial F_t}{\partial \underline{\phi}} = 2 S^{(t)} \underline{\phi} - 2\, \underline{b}^{(t)}$$

$$\boxed{S^{(t)} \underline{\phi} = \underline{b}^{(t)}}$$

$$S^{(t)} = \frac{1}{4A_e^{(t)}} \begin{bmatrix} \nabla\alpha_i \cdot \nabla\alpha_i & \nabla\alpha_i \cdot \nabla\alpha_j & \nabla\alpha_i \cdot \nabla\alpha_m \\ \nabla\alpha_j \cdot \nabla\alpha_i & \nabla\alpha_j \cdot \nabla\alpha_j & \nabla\alpha_j \cdot \nabla\alpha_m \\ \nabla\alpha_m \cdot \nabla\alpha_i & \nabla\alpha_m \cdot \nabla\alpha_j & \nabla\alpha_m \cdot \nabla\alpha_m \end{bmatrix}^{(t)}$$

$$= \frac{1}{4A_e^{(t)}} \begin{bmatrix} b_i b_i + c_i c_i & b_i b_j + c_i c_j & b_i b_m + c_i c_m \\ b_j b_i + c_j c_i & b_j b_j + c_j c_j & b_j b_m + c_j c_m \\ b_m b_i + c_m c_i & b_m b_j + c_m c_j & b_m b_m + c_m c_m \end{bmatrix}^{(t)}$$

$$\underline{b}^{(t)} = \int_{\Omega^t} \begin{pmatrix} \alpha_i \\ \alpha_j \\ \alpha_m \end{pmatrix} f(x,y)\, dx\, dy$$

if $\quad f(x,y) = \rho/\varepsilon \quad$ (constant) $\quad$ then

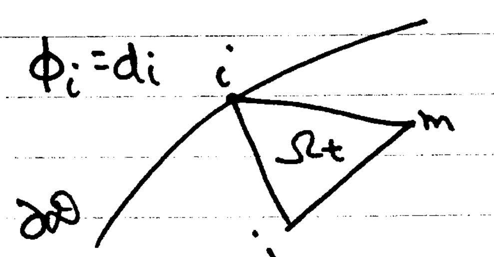
$$\underline{b}^{(t)} = \frac{1}{3\varepsilon} \begin{pmatrix} \rho A_e^{(t)} \\ \rho A_e^{(t)} \\ \rho A_e^{(t)} \end{pmatrix} \leftarrow \text{element Area } /3$$

short-hand notation:
$$\begin{cases} S_{ij} = \dfrac{1}{4A_e^{(t)}} (b_i b_j + c_i c_j) \\ b_i = \dfrac{\rho A_e^{(t)}}{3\varepsilon} \end{cases}$$

∴ the contribution of $S^{(t)}$ in the global matrix S given that the nodes of triangle t are i, j, m is as follows:

$$
\begin{array}{c c}
 & \begin{array}{ccccccc} \cdots & i & \cdots & j & \cdots & m & \cdots \end{array} \\
\begin{array}{c} \vdots \\ i \\ \vdots \\ j \\ \vdots \\ m \\ \vdots \end{array} &
\begin{bmatrix}
 & \vdots & & \vdots & & \vdots & \\
\cdots & S_{ii}^{(t)} & \cdots & S_{ij}^{(t)} & \cdots & S_{im}^{(t)} & \cdots \\
 & \vdots & & \vdots & & \vdots & \\
\cdots & S_{ji}^{(t)} & \cdots & S_{jj}^{(t)} & \cdots & S_{jm}^{(t)} & \cdots \\
 & \vdots & & \vdots & & \vdots & \\
\cdots & S_{mi}^{(t)} & \cdots & S_{mj}^{(t)} & \cdots & S_{mm}^{(t)} & \cdots \\
 & \vdots & & \vdots & & \vdots &
\end{bmatrix}
\end{array}
\begin{bmatrix} \phi_1 \\ \vdots \\ \phi_i \\ \vdots \\ \phi_j \\ \vdots \\ \phi_m \\ \vdots \\ \phi_N \end{bmatrix}
=
\begin{bmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_j \\ \vdots \\ b_m \\ \vdots \\ b_N \end{bmatrix}
$$

② triangle with a node on a Dirichlet boundary:



Since ϕ_i is known it is not a free node and ∴ not a variational parameter

For the element:

$$
\begin{cases}
S_{ji}^{(t)} d_i + S_{jj}^{(t)} \phi_j + S_{jm}^{(t)} \phi_m = b_j \\
S_{mi}^{(t)} d_i + S_{mj}^{(t)} \phi_j + S_{mm}^{(t)} \phi_m = b_m
\end{cases}
$$

$$
\therefore \quad
\begin{cases}
S_{jj}^{(t)} \phi_j + S_{jm}^{(t)} \phi_m = b_j - S_{ji} d_i \\
S_{mj}^{(t)} \phi_j + S_{mm}^{(t)} \phi_n = b_m - S_{mi}^{(t)} d_i
\end{cases}
$$

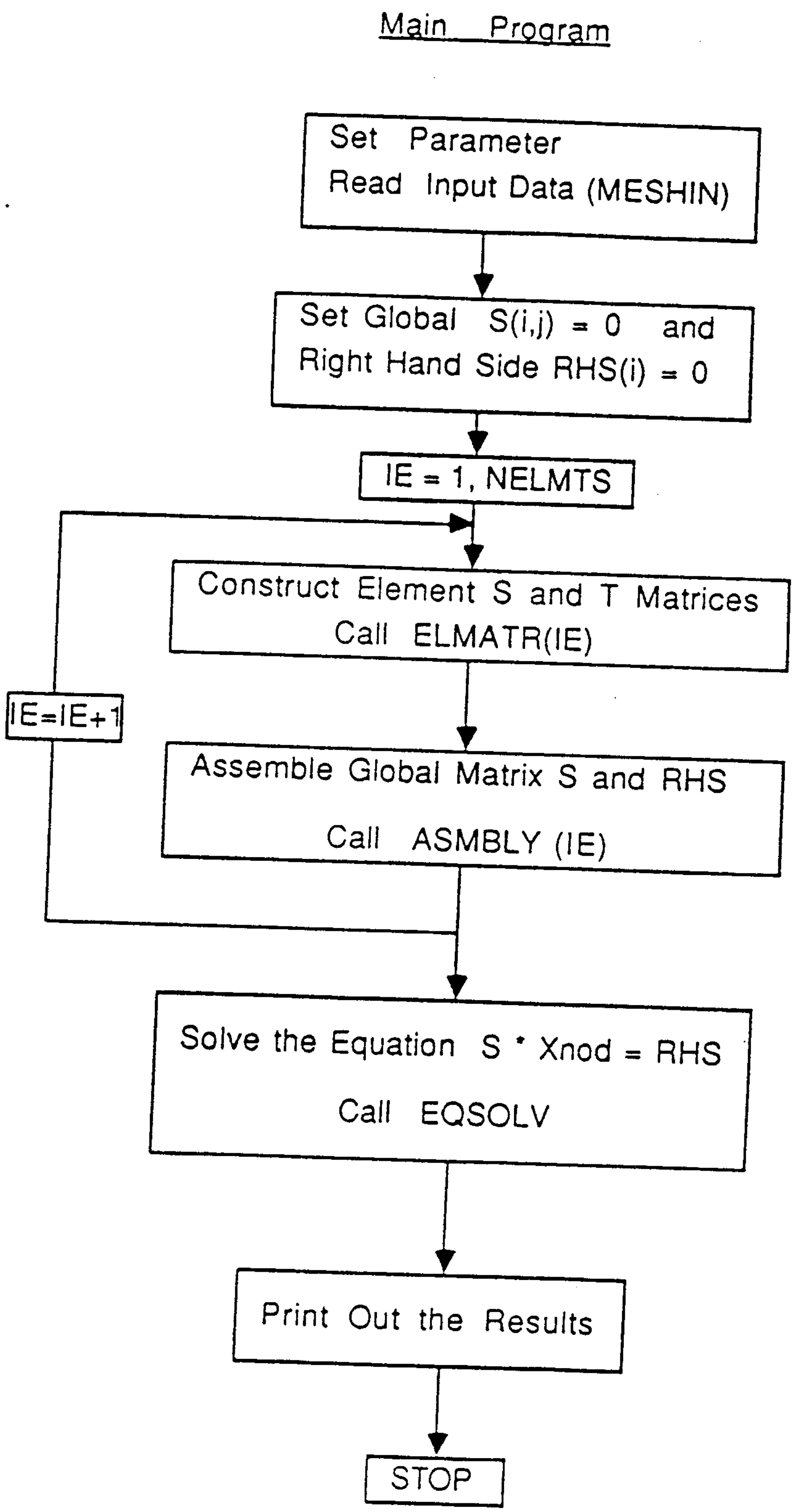
Notes:

1) Nodes on Dirichlet B.C.'s contribute to the right-hand side of the final matrix equation.

2) Neumann B.C.'s are called "natural": any node on a boundary which is not forced will approximate a Neumann B.C.

3) The final matrix is symmetric

Sample computer program



Subroutine ASMBLY